Soft Collateral, Bank Lending, and the Optimal Credit Rating System

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Abstract

In this paper, we study the optimal credit rating system in an economy where infinitely-lived agents need to borrow and have incentives to renege on debt repayments. We show that credit exclusion is a form of soft collateral. Compared with individual lending, bank lending reduces search frictions, whereby increasing the cost of credit exclusion, boosting the value of soft collateral, and facilitating borrowing and lending. A dynamic rating system allows agents’ ratings to migrate over time and fine-tunes agents’ incentives. By doing so, it reduces the agency cost, makes better use of soft collateral, and improves social welfare. We show that the optimal rating system is coarse, as we observe in the real world.

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1 Introduction

The conventional wisdom is that a credit rating system is a way to evaluate credit worthiness of a potential borrower—a person, a corporation, a local government, or a country—who may not honor the commitment to repay the loan. There are two reasons why a borrower may renege on its contractual obligations: it does not have the ability to fulfill its obligations or it is able but unwilling to fulfill its obligations. As a result, credit ratings measure both the ability and the willingness. Unlike information about ability, information about willingness is related to moral hazard, that is, a borrower’s incentive to honor the contractual obligations. While most of the studies focus on the function of credit ratings in providing information about a borrower’s ability, this paper focuses on information about a borrower’s incentive. We want to make a point that, in addition to providing information about a borrower’s past, credit ratings can predict a borrower’s behavior in the future. This predictive power has its social benefit as well as social cost. We show that the trade-off between the social benefit and social cost determines that the optimal rating system is “coarse” in the sense that it only consists of a finite number of ratings, as we observe in the real world.

Our paper is closely related to the literature on borrowers’ reputation and its disciplinary effect. Diamond (1989) illustrates that reputation can be used to alleviate the conflict of interest between borrowers and lenders. In his model, there are three types of borrowers—good ones, bad ones, and ones who can choose to be good or bad. He shows that the third type borrowers’ moral hazard problem can be corrected with reputation. As time goes on, the reputation value goes up and older borrowers will be incentivised to choose good projects. Contrary to Diamond (1989), Vercammen (1995) shows that if, unlike what happens in Diamond (1989), bad borrowers are never excluded from the market, then the reputation effect can decreases as lenders learn more and more about borrowers over time. Padilla and Pagano (2000) study a two period model and show that information sharing between banks can mitigate the moral hazard problem: in order to avoid being pooled with low-quality borrowers, high-quality borrowers work hard to avoid default. Similar to these papers, we
focus on information about borrowers’ actions rather than qualities (for papers on borrowers’ qualities, see, for example, Diamond (1991), Pagano and Jappelli (1993), and Padilla and Pagano (1997)). We share the common idea with other researchers that borrowers’ reputation is a disciplinary device for addressing moral hazard. The major contributions of our paper is that it generates several key features that are in line with what happens in the real world. First, there is a steady state cross-sectional distribution of credit ratings at any given time. Second, contrary to the monotonic convergence of reputation predicted by other papers, agents’ ratings can migrate over time in our paper. Third, unlike other papers that focus on interest rate differentiation, we show that interest rate differentiation cannot substitute for credit exclusion to discipline borrowers. Fourth, the optimal credit rating system does not exclude defaulted borrowers permanently; instead, as what occurs in the real world, it allows excluded borrowers to return to the capital market. Fifth and finally, we show that, consistent with rating systems in the real world, the optimal rating system is coarse.

Specifically, we build a model where a borrower’s ability to repay the debt is private information and borrowers have incentives to default strategically. We show that access to future credit is a form of soft collateral that lenders hold against borrower’s loans. Accordingly, excluding defaulted borrowers from the capital market is a necessary social cost to prevent borrowers from defaulting strategically. Without banks, the capital market is inefficient because dispersed individual lenders cannot effectively exclude defaulted borrowers. Banks play a very important role in providing a centralized loan market. Centralization reduces search frictions, increases the cost of default, boosts the value of soft collateral, and facilitates borrowing and lending. However, the large number of banks in a competitive banking system creates information inefficiency—a borrower who reneges on loan repayment may not be blacklisted by all other banks. This inefficiency calls for a shared information system—a credit rating system. The function of credit ratings goes beyond information efficiency, especially in a dynamic model. A multi-tier rating system makes better use of information, creates a better structure of soft collateral, and reduces the necessity for credit exclusion.
We show that the optimal credit rating system is coarse, in the sense that it consists of only a finite number of ratings. A system with more tiers of ratings excludes a defaulted borrower less frequently, but, to satisfy the incentive compatibility condition, it needs to punish defaulted borrowers more severely by excluding them from the capital market for a longer time before allowing them to get a fresh new start. In equilibrium, the optimal credit rating system balances the frequency and severity of punishment and reduces the social cost to the minimum.

There is a growing literature on credit ratings. Early work in this area views rating agencies as agents who specialize in producing information and selling it to other people; the focus is generally on risk diversification and information sharing (for example, see Ramakrishnan and Thakor (1984), Millon and Thakor (1985), Allen (1990), and Fishman and Hagerty (1995)). Related to this line of thought, more recent work considers whether agents have incentives to transmit distorted information to the market. For example, Opp, Opp and Harris (2013) show that regulators’ use of the rating information might decrease rating informativeness; Fulghieri, Strobl and Xia (2013) study a dynamic rational expectations model to show that, in order to extract higher fees, rating agencies have incentives to make unsolicited credit ratings unfavorable. There are also papers that examine the competition between rating agencies and its effect on social welfare. Skreta and Veldkamp (2009) study a rating shopping model to examine the effect of asset complexity on rating inflation. Bolton, Friexas, and Shapiro (2012) show that competition induces rating shopping, which leads to rating inflation and reduces efficiency. Sangiorgi and Spatt (2015) consider a model where issuers purchase ratings sequentially, and information asymmetry causes selective disclosure and rating bias.\footnote{For other papers on credit ratings, see Boot, Milbourn, and Schmeits (2006), Mathis, McAndrews and Rochet (2009), Manso (2013).}

Our paper is more closely related to works on rating coarseness. Lizzeri (1999) shows that a monopoly intermediary, in order to maximize the surplus, has an incentive to manipulate information by revealing only whether quality is above some minimal standard. Competition
among the intermediaries can force them to reveal full information. Goel and Thakor (2013) construct a cheap-talk game to model coarse ratings. In equilibrium, a rating agency wants to deliver inflated ratings to please issuers, and, in the meantime, needs to keep the rating inflation below a threshold to make it credible to investors. The two conflicting objectives give rise to coarse but unbiased ratings in equilibrium. Coarse ratings reduce social welfare because they lead to investment inefficiency. Kovbasyuk (2013) shows that private rating-contingent payments can cause ratings coarseness. Kartasheva and Yilmaz (2013) show that ratings become less precise when there are more uninformed investors in the market and the gains of trade increase. Donaldson and Piacentino (2013) consider credit ratings as a source of public information and show that a reduction in rating precision can Pareto improve social welfare. Our paper is different in that, instead of considering rating agencies’ incentives and the relative advantage of private information, we focus on the effect of ratings on borrower’s incentives. An optimal rating system has to be coarse because it needs to satisfy incentive compatibility constraints of agents with various ratings. There is no room for regulators to improve efficiency in our framework.

The rest of the paper proceeds as follows. In Section 2, we set up the model and lay out the assumptions. In Section 3, we first study the autarky case where there is no borrowing and lending; we then analyze the capital market without banks where borrowing and lending can only occur through random matching of dispersed individuals. We examine the centralized bank lending market in section 4. We investigate credit ratings in Section 5. We first study a simple three-tier rating system to illustrate the intuition; afterwards we solve the general multi-tier rating equilibrium and characterize the optimal rating system. Section 6 includes discussions and Section 7 offers conclusive remarks. The appendix includes the proofs of propositions.
2 Model

The economy is populated with a continuum of infinitely lived agents, with the total population normalized to unity. Agents produce and consume perishable goods at discrete points in continuous time. Each period, a fraction \( c \in [0, 1] \) of the population are each endowed with one (normalized) unit of capital, which is needed to produce consumption goods. In addition, all agents, with or without capital, receive a productivity shock that is independent of the capital endowment shock. With probability \( p \), an agent’s productivity is high \((H)\); with probability \( 1 - p \), his productivity is low \((L)\). We assume that the distribution of capital and productivity shocks are independent and identical over time. So, conditional on capital endowment and productivity shocks, each period there are four types of agents in the economy: those with capital and high productivity, whose value function denoted by \( V_{1H} \); those with capital but low productivity, whose value function denoted by \( V_{1L} \); those with high productivity but no capital, whose value function denoted by \( V_{0H} \); and those with low productivity and no capital, whose value function denoted by \( V_{0L} \).

With one unit of capital, an agent with high productivity produces random output: either \( X \) units of the consumption good with probability \( \pi \) or zero consumption good with probability \( 1 - \pi \); the expected output is \( X_H = \pi X \). We assume that \( X_H \) is greater than a low-productivity agent’s output per unit of capital, which, for simplicity, is assumed to be a constant \( X_L \). We assume that a high-productivity agent’s realized output is neither observable nor verifiable, which gives rise to the moral hazard problem, the solution to which is the key point of the paper. We also assume that capital goods cannot be consumed and are used up in the process of producing consumption goods, so there is no capital accumulation. In addition, capital goods are indivisible and each agent can only use one unit of capital. All agents are risk neutral, and the discount rate is \( \beta \) per period. We first study the equilibrium in the absence of financial intermediaries.
3 Equilibrium without Financial Intermediaries

In this section, we analyze the equilibrium in an economy where there is no financial intermediary. We first solve the autarky case, then consider the case where individual borrowing and lending are allowed.

3.1 Autarky

We first solve the value functions for the case of autarky where there is no borrowing and lending. In this case, we have:

\[ V_{1H}^A = \frac{1}{1 + \beta} \{ X_H + V^A \}, \]
\[ V_{1L}^A = \frac{1}{1 + \beta} \{ X_L + V^A \}, \]
\[ V_{0H}^A = \frac{V^A}{1 + \beta}, \]
\[ V_{0L}^A = \frac{V^A}{1 + \beta}, \]

where

\[ V^A = c p V_{1H}^A + c (1 - p) V_{1L}^A + (1 - c) p V_{0H}^A + (1 - c) (1 - p) V_{0L}^A \]

is the unconditional expected lifetime value at the beginning of a period, before each agent knows whether he will receive capital in the period and whether his productivity is high or low. The following proposition describes the autarky equilibrium:

**Proposition 1** In the autarky equilibrium, an agent’s expected lifetime payoff is equal to

\[ \frac{c p X_H + c (1 - p) X_L}{\beta}. \]

**Proof.** See Appendix. ■

The proposition is easy to interpret. Each period an agent receives capital with probability \( c \), and, with the capital endowment, produces \( X_H \) with probability \( p \) and \( X_L \) with
probability $1 - p$. Therefore, the expected payoff is $cpX_H + c(1 - p)X_L$. The ex ante unconditional expected lifetime value is just a perpetuity of periodical payoffs with the expected value equal to $cpX_H + c(1 - p)X_L$. Ex post, if an agent does not own capital, he receives nothing during the current period, and thus the lifetime value is the perpetuity postponed by one period; discounted by $1 + \beta$, it is $\frac{V^A}{1 + \beta}$. If an agent owns capital in the current period, then in addition to the postponed perpetuity, he is going to receive $X_H$ or $X_L$ at the end of the current period depending on whether his productivity is high or low. Since agents are homogeneous, social welfare in the autarky economy is the same as an agent’s unconditional expected lifetime value:

$$W^A = V^A = \frac{cpX_H + c(1 - p)X_L}{\beta}.$$ 

The autarky economy is inefficient because a fraction of capital is stuck in the hands of those agents with low productivity while some of the high-productivity agents do not have access to the indispensable capital for production. The inefficiency calls for a financial market where agents can borrow and lend capital to generate more outputs. In the remaining of this paper, we analyze financial markets that allow borrowing and lending, starting with individual loans, then bank loans, and finally, bank loans with credit ratings.

### 3.2 Individual Loans

In this section, we consider the case of a decentralized market with individual loans. We assume agents randomly meet after capital and productivity shocks are realized. Borrowing and lending happen only when a capital owner with low productivity meets an agent with high productivity but no capital; the former then becomes a capital borrower and the latter becomes a capital lender. Considering the overall distribution of different agent types, a borrower meets a lender with probability $c(1 - p)$, and a lender meets a borrower with probability $(1 - c)p$. A borrower agrees to pay $R$ to the lender at the end of the period after production is completed. Because production is risky, the lender has a chance to receive $R$
only when a high-productivity borrower generates \( X \) units of the consumption good; this happens with probability \( \pi \). Moreover, because output is neither observable nor verifiable, without any potential punishment, the borrower has no incentive to repay the debt.

The punishment for default is credit exclusion. Specifically, we assume that with probability \( \gamma \) a defaulted borrower obtains a bad reputation and will be denied of loans from any other agent in the next period. However, reputation can be repaired. After one period, with probability \( \eta \) a defaulted agent will get a fresh start and be able to borrow again; with probability \( 1 - \eta \), the bad reputation sticks and the default agent has to wait for one more period to see whether he has a chance to be allowed to borrow. In the steady state, a fraction \( \alpha^I \) of the population do not have the bad reputation; their value functions conditional on realized capital and productivity shocks are as follows:

\[
V_{1H}^I = \frac{1}{1 + \beta} \{X_H + V^I\}, \\
V_{1L}^I = \frac{1}{1 + \beta} \{(1 - c)p\pi R + (1 - (1 - c)p)X_L + V^I\}, \\
V_{0H}^I = \frac{1}{1 + \beta} \{c(1 - p)[(1 - \pi)((1 - \gamma)V^I + \gamma V_{e}^I)] + (1 - c(1 - p))V^I\}, \\
V_{0L}^I = \frac{V^I}{1 + \beta},
\]

where

\[
V^I \equiv cpV_{1H}^I + c(1 - p)V_{1L}^I + (1 - c)pV_{0H}^I + (1 - c)(1 - p)V_{0L}^I
\]

is the unconditional expected lifetime value at the beginning of a period before capital and productivity shocks are realized. As for those agents with the bad reputation, the remaining \( 1 - \alpha^I \) fraction of the population, we denote their unconditional expected lifetime value by \( V_{e}^I \),

\[
V_{e}^I = \frac{1}{1 + \beta} \{cpX_H + c(1 - p)X_L + \eta V^I + (1 - \eta)V_{e}^I\}.
\]

The equilibrium solutions of the value functions are subject to the incentive compatibility conditions:
1). Lenders are willing to lend:
\[ \pi R \geq X_L; \]

2). Borrowers with high outputs are willing to repay the loan:
\[ R \leq \gamma(V^I - V^I_e); \]

3). A constant steady state population distribution:
\[ \alpha^I c(1 - c)p(1 - p)(1 - \pi)\gamma = \eta(1 - \alpha^I). \]

We assume that borrowers have all the bargaining power, so \( R = X_L / \pi \). A borrower chooses between repaying the loan and facing the punishment of potential credit exclusion next period. Credit exclusion essentially serves as a form of soft collateral. The likelihood of being excluded from the loan market, \( \gamma \), has a direct effect on the value of the soft collateral; in addition, together with other model parameters, it has an indirect effect on the value of the soft collateral through its impact on \( V^I - V^I_e \). A borrower repays the debt if and only if the value of the soft collateral exceeds the gain from strategic default. Solving the model, we have:

Proposition 2 There exists a private loan market if and only if: 1) the likelihood of excluding a defaulted borrower from the loan market is large enough: given all the other parameters, there is a minimum value of \( \gamma \), \( \gamma^I \), such that \( \gamma \geq \gamma^I \); or 2) the chance of returning to the loan market is small enough: given all the other parameters, there is a maximum value of \( \eta \), \( \eta^I \), such that \( \eta \leq \eta^I \).

Proof. See Appendix. ■

Proposition 2 shows that the existence of a private loan market depends on the value of the soft collateral, which is determined by likelihood of blackballing a defaulted borrower.
Social welfare in this case is equal to the weighted average of the expected lifetime value:

\[
W^I = \alpha V^I + (1 - \alpha)V^I_e \\
= \frac{c p X_H + c (1 - p) X_L}{\beta} + \frac{c (1 - c) p (1 - p) (X_H - X_L)}{\beta [1 + c (1 - c) p (1 - p) (1 - \pi) \gamma / \eta]}
\]

As can be seen, social welfare is decreasing in \( \gamma \) and increasing in \( \eta \). Being excluded from the loan market, defaulted borrowers cannot take advantage of their high productivity, but this welfare loss is the necessary cost to guarantee that borrowers have incentives to repay the debt.

A decentralized private loan market faces two obstacles that hamper the value of the soft collateral and prohibit borrowing and lending. First, it is very difficult to share information about a borrower’s default and to exclude him from the loan market; in other words, \( \gamma \) is small and \( \eta \) can be large. Second, search frictions limit the chance of meeting a lender and thus softens the punishment of being excluded from the loan market. As a result, a private loan market can only exist in a closely-knit community where people are familiar with each other and information is more transparent. We proceed to show that bank lending, even with the same parameter values of \( \gamma \) and \( \eta \), can boost the value of the soft collateral.

4 Bank Loans

Suppose there is a competitive banking system in which banks accept deposits from agents who are endowed with capital and low productivity, and make loans to agents who have high-productivity but lack capital. The existence of competitive banks alleviates the double coincidence problem because borrowers and depositors do business directly with banks instead of meeting each other through random matching. As a result, it is more costly for a borrower to default and be excluded from the loan market.

We assume that depositors receive \( R_d \) by saving their capital goods with banks; borrowers who receive bank loans agree to pay \( R_l \) at the end of the period. In the steady state, a
fraction $\alpha^B$ of all the agents are allowed to borrow from banks and the remaining $1 - \alpha^B$ of all the agents are excluded from borrowing for at least one period due to default in the past. Agents who are not blacklisted by banks have the following value functions once the capital and productivity shocks are realized:

$$V_{1H}^B = \frac{1}{1+\beta} \{X_H + V^B\},$$
$$V_{1L}^B = \frac{1}{1+\beta} \{R_d + V^B\},$$
$$V_{0H}^B = \frac{1}{1+\beta} \{\pi(X - R_l + V^B) + (1 - \pi)(1 - \gamma) V^B + \gamma V_e^B\},$$
$$V_{0L}^B = \frac{V^B}{1+\beta},$$

where

$$V^B \equiv cpV_{1H}^B + c(1-p)V_{1L}^B + (1-c)pV_{0H}^B + (1-c)(1-p)V_{0L}^B$$

is the unconditional expected lifetime value at the beginning of a period. Agents who are blacklisted by banks have the following unconditional expected lifetime value function:

$$V_e^B = \frac{1}{1+\beta} \{cpX_H + c(1-p)X_L + \eta V^B + (1-\eta)V_e^B\}.$$  

In equilibrium, the following constraints need to be satisfied:

1). Depositors are willing to put their capital into banks:

$$R_d \geq X_L;$$

2). Borrowers with high outputs are willing to repay bank loans:

$$R_l \leq \gamma(V^B - V_e^B);$$
3). Banks break even:
\[ \pi R_l \geq R_d; \]

4). A constant steady state population distribution:
\[ \alpha^B(1 - c)p(1 - \pi)\gamma = \eta(1 - \alpha^B). \]

We still assume that the overall supply of deposits is greater than the demand for loans. As a result, banks will complete to lower the deposit rate and the loan rate such that we have \( R_d = X_L \) and \( \pi R_l \geq X_L \) in equilibrium. Proposition 3 characterizes the equilibrium solutions:

**Proposition 3** Compared with the private loan economy, a competitive banking system improves economic efficiency. Specifically, let \( \gamma^B (\eta^B) \) denote the minimum (maximum) value of \( \gamma (\eta) \), ceteris paribus, for a bank loan equilibrium to exist. We have \( \gamma^B < \gamma^I \) and \( \eta^B > \eta^I \); that is, when \( \gamma^B \leq \gamma < \gamma^I \) (or \( \eta^B \geq \eta > \eta^I \)), there exists a bank loan equilibrium but not a private loan equilibrium. In addition, when \( \gamma \geq \gamma^I \) (or \( \eta \leq \eta^I \)), a bank loan equilibrium is always more efficient than a private loan equilibrium.

**Proof.** See Appendix \( \blacksquare \)

With banks existing in the economy, borrowers know where exactly to obtain capital to exploit their high productivity and will always get it if they are not blacklisted by banks. In contrast, because of search frictions, a borrower with good reputation only obtains capital with probability \( c(1 - p) \)– the agent he meets is endowed with capital and low productivity–in the private loan economy. Proposition 3 shows that the reduction of search frictions has a huge impact beyond itself because it greatly increases the cost of credit exclusion, and, by doing so, it increases the value of the soft collateral. Consequently, a small chance of being blacklisted by banks can become a huge cost for defaulted borrowers. The tightened incentive relaxes constraints on parameters and improves social welfare.
What is worth mentioning is that, although concentrated lending makes it easier to blacklist defaulted borrowers—that is, bank lending is presumably associated with a higher $\gamma$ and a lower $\eta$, this is not the source of improved efficiency; instead, if anything, it is a source of inefficiency. We only need the parameter values of $\gamma$ and $\eta$ to guarantee the existence of the bank loan equilibrium; beyond those values, a higher $\gamma$ or a lower $\eta$ reduces social welfare.

So far we have shown that a competitive banking system is more efficient than a private loan economy, but can it be further improved? In a dynamic model as we study in this paper, each agent has a long history of transactions. Because all agents are homogeneous at the very beginning, when the history is long enough, each agent’s history has essentially the same frequency of borrowing, lending, repayments, and defaults; that is, agents’ credit quality is statistically indistinguishable. Even so, we can create a rating system based on a truncated history—for example, the most recent transaction—to distinguish agents. The rational is that a rating system allows a multiple-tier punishment scheme so that some defaulted borrowers get their ratings downgraded, a probation in a sense, instead of being immediately excluded from the loan market. Designed properly, downgrading can give borrowers incentives to repay their loans, and it is a less costly solution to the moral hazard problem compared with credit exclusion. We investigate credit ratings in the next section.

5 Credit Ratings

To understand how a rating system contributes to social welfare, we first analyze a simple case where each agent is assigned one of the three ratings: $A$, $B$, or $E$. Afterwards, we extend our analysis to a general system with $N$ ratings and characterize the optimal rating system.
5.1 A Three-tier Rating System

We extend the analysis in Section 5 by further dividing those agents who are not excluded from borrowing into two subgroups: \( A \) and \( B \). So at the beginning of each period, before capital and productivity shocks are realized, each agent has one of the three ratings: \( A \), \( B \), or \( E \). If an agent with rating \( A \) borrows and defaults, then his rating is downgraded to \( B \); otherwise he keeps the original rating \( A \). If an agent with rating \( B \) borrows and repays the loan, his rating is upgraded to \( A \); if he borrows and defaults, then his rating is downgraded to \( E \) with probability \( \gamma \); in all other cases he keeps the original rating \( B \). An agent with rating \( E \) is excluded from borrowing in the current period but has a chance to be upgraded to rating \( B \) next period, which happens with probability \( \eta \); with probability \( 1 - \eta \), he remains the original rating \( E \) next period. We use superscripts \( RA \) and \( RB \) to differentiate agents with ratings \( A \) and \( B \) respectively; as for agents with rating \( E \), we still use the subscript \( e \) to denote them.

Agents with rating \( A \) have the following value functions once the capital and productivity shocks are realized:

\[
V_{1H}^{RA} = \frac{1}{1 + \beta} \{X_H + V^{RA}\}, \\
V_{1L}^{RA} = \frac{1}{1 + \beta} \{R_d + V^{RA}\}, \\
V_{0H}^{RA} = \frac{1}{1 + \beta} \{\pi(X - R_l + V^{RA}) + (1 - \pi)V^{RB}\}, \\
V_{0L}^{B} = \frac{V^{RA}}{1 + \beta},
\]

where

\[
V^{RA} \equiv cpV_{1H}^{RA} + c(1 - p)V_{1L}^{RA} + (1 - c)pV_{0H}^{RA} + (1 - c)(1 - p)V_{0L}^{RA}
\]

is the unconditional expected lifetime value at the beginning of a period. Agents with rating
$B$ have the following value functions once the capital and productivity shocks are realized:

\[
V_{1H}^{RB} = \frac{1}{1 + \beta} \{X_H + V_{RB}^c\},
\]

\[
V_{1L}^{RB} = \frac{1}{1 + \beta} \{R_d + V_{RB}^c\},
\]

\[
V_{0H}^{RB} = \frac{1}{1 + \beta} \{\pi (X - R_t + V^{RA}) + (1 - \pi)[(1 - \gamma)V_{RB}^c + \gamma V_{e}^R]\},
\]

\[
V_{0L}^{RB} = \frac{V_{RB}^c}{1 + \beta},
\]

where

\[
V_{RB}^c \equiv cpV_{1H}^{RB} + c(1 - p)V_{1L}^{RB} + (1 - c)pV_{0H}^{RB} + (1 - c)(1 - p)V_{0L}^{RB}
\]

is the unconditional expected lifetime value at the beginning of a period.

Agents with rating $E$ have the following unconditional expected lifetime value:

\[
V_{e}^R = \frac{1}{1 + \beta} \{cpX_H + c(1 - p)X_L + \eta V_{RB}^c + (1 - \eta)V_{e}^R\}.
\]

The value functions are subject to the following constraints:

1). Depositors are willing to deposit their capital in banks:

\[
R_d \geq X_L;
\]

2a). Borrowers with rating $A$ are willing to repay bank loans:

\[
R_t \leq V^{RA} - V^{RB};
\]

2b). Borrowers with rating $B$ are willing to repay bank loans:

\[
R_t \leq V^{RA} - [\gamma V_{e}^R + (1 - \gamma)V^{RB}].
\]
3). Banks break even:

\[ \pi R_l \geq R_d; \]

4). A constant steady state population distribution:

\[ \alpha^{RA}(1 - c)p(1 - \pi) = \alpha^{RB}(1 - c)p\pi, \]

\[ \alpha^{RB}(1 - c)p(1 - \pi)\gamma^B = (1 - \alpha^{RA} - \alpha^{RB})\eta^B, \]

where \( \alpha^{RA} \) and \( \alpha^{RB} \) denote the proportion of agents with ratings A and B respectively.

Same as before, we assume that competition drives the deposit rate and the loan rate to the minimum level; that is, \( R_d = X_L \) and \( \pi R_l = X_L \).

**Proposition 4** Let \( \overline{\gamma}^R (\overline{\eta}^R) \) denote the minimum (maximum) value of \( \gamma (\eta) \), ceteris paribus, for a bank loan equilibrium with credit ratings to exist. We have \( \overline{\gamma}^R > \gamma^B \) and \( \overline{\eta}^R < \overline{\eta}^B \); that is, when \( \gamma^B \leq \gamma < \gamma^R \) (or \( \overline{\eta}^B \geq \eta > \overline{\eta}^R \)), there exists a bank loan equilibrium without credit ratings but not a bank loan equilibrium with credit ratings. However, when \( \gamma \geq \gamma^R \) (or \( \eta \leq \overline{\eta}^R \)), a bank loan equilibrium with credit ratings is always more efficient than that without credit ratings.

**Proof.** See Appendix. □

Proposition 4 tells us that, so long as parameter values allow the three-tier credit rating system to exist, it is always more efficient than a banking system without credit ratings. Credit ratings reduce the social cost by giving some of the defaulted borrowers—those with the rating A—a second chance rather than immediately excluding them from borrowing. By doing so, credit ratings require two tiers of punishment: downgrading from A to B and downgrading from B to E. To discourage borrowers from strategic default, the costs of being downgraded in both cases need to be greater than the amount of loan repayment. This requires a minimum aggregate gap between the value of rating A and that of rating E, which can only be guaranteed with a significant threat to exclude defaulted borrowers—a
higher cutoff value of $\gamma$ or a lower cutoff value of $\eta$ compared with the cutoff values in a banking system without credit ratings.

### 5.2 A General Rating System

In this subsection, we extend the analysis to a general rating system that consists of $N$ different ratings, indexed as $1, 2, \ldots, N-1, N(E)$, from the best to the worst. If an agent borrows and repays the loan, then his rating is upgraded one level above except for agents with rating 1, who keep the original rating 1. If an agent borrows and defaults, then his rating is downgraded one level except for agents with ratings $N-1$, who is downgraded to $E$ with probability $\gamma$. An agent with rating $E$ is denied of borrowing in the current period but has a chance to be upgraded to rating $N-1$ next period, which happens with probability $\eta$; with probability $1-\eta$, he remains the original rating $E$ next period. We use superscripts $G(k)$ ($k = 1, 2, \ldots, N-1$) to differentiate agents with ratings $1, 2, \ldots, N-1$; as for agents with rating $N(E)$, we still use the subscript $e$ to denote them.

Agents with rating 1 have the following value functions once the capital and productivity shocks are realized:

\[
V_{1H}^{G(1)} = \frac{1}{1+\beta} \{X_H + V_{1H}^{G(1)}\}, \\
V_{1L}^{G(1)} = \frac{1}{1+\beta} \{R_d + V_{1L}^{G(1)}\}, \\
V_{0H}^{G(1)} = \frac{1}{1+\beta} \{\pi(X - R_l + V_{0H}^{G(1)}) + (1-\pi)\gamma V_{0H}^{G(2)}\}, \\
V_{0L}^{G(1)} = \frac{V_{0L}^{G(1)}}{1+\beta},
\]

where

\[
V_{1H}^{G(1)} = cpV_{1H}^{G(1)} + c(1-p)V_{1L}^{G(1)} + (1-c)pV_{0H}^{G(1)} + (1-c)(1-p)V_{0L}^{G(1)}
\]

is the unconditional expected lifetime value at the beginning of a period.

Agents with rating $k$ ($k = 2, 3, \ldots, N-2$) have the following value functions once the
capital and productivity shocks are realized:

\[
\begin{align*}
V_{1H}^{G(k)} &= \frac{1}{1 + \beta} \{X_H + V_{1H}^{G(k)}\}, \\
V_{1L}^{G(k)} &= \frac{1}{1 + \beta} \{R_d + V_{1L}^{G(k)}\}, \\
V_{0H}^{G(k)} &= \frac{1}{1 + \beta} \{\pi(X - R_t + V_{0H}^{G(k-1)}) + (1 - \pi)V_{0H}^{G(k+1)}\}, \\
V_{0L}^{G(k)} &= \frac{V_{0L}^{G(k-1)}}{1 + \beta},
\end{align*}
\]

where

\[V^{G(k)} \equiv cpV_{1H}^{G(k)} + c(1 - p)V_{1L}^{G(k)} + (1 - c)pV_{0H}^{G(k)} + (1 - c)(1 - p)V_{0L}^{G(k)}\]

is the unconditional expected lifetime value at the beginning of a period.

Agents with rating \(N - 1\) have the following value functions once the capital and productivity shocks are realized:

\[
\begin{align*}
V_{1H}^{G(N-1)} &= \frac{1}{1 + \beta} \{X_H + V_{1H}^{G(N-1)}\}, \\
V_{1L}^{G(N-1)} &= \frac{1}{1 + \beta} \{R_d + V_{1L}^{G(N-1)}\}, \\
V_{0H}^{G(N-1)} &= \frac{1}{1 + \beta} \{\pi(X - R_t + V_{0H}^{G(N-2)}) + (1 - \pi)[(1 - \gamma)V_{0H}^{G(k-1)} + \gamma V_{e}^{G}]\}, \\
V_{0L}^{G(N-1)} &= \frac{V_{0L}^{G(N-1)}}{1 + \beta},
\end{align*}
\]

where

\[V^{G(N-1)} \equiv cpV_{1H}^{G(N-1)} + c(1 - p)V_{1L}^{G(N-1)} + (1 - c)pV_{0H}^{G(N-1)} + (1 - c)(1 - p)V_{0L}^{G(N-1)}\]

is the unconditional expected lifetime value at the beginning of a period.

Finally, agents with rating \(E\) have the following unconditional expected lifetime value:

\[V_{e}^{G} = \frac{1}{1 + \beta} \{cpX_H + c(1 - p)X_L + \eta V_{e}^{G(N-1)} + (1 - \eta)V_{e}^{G}\}.
\]
The value functions are subject to following constraints:

1). Depositors are willing to deposit their capital in banks:

\[ R_d \geq X_L; \]

2a). Borrowers with rating 1 are willing to repay bank loans:

\[ R_l \leq V^{G(1)} - V^{G(2)}; \]

2b). Borrowers with rating \( k \) \((k = 2, 3, \ldots N - 2)\) are willing to repay bank loans:

\[ R_l \leq V^{G(k-1)} - V^{G(k+1)}; \]

2c). Borrowers with rating \( N - 1 \) are willing to repay bank loans:

\[ R_l \leq V^{G(N-2)} - [\gamma V_e^G + (1 - \gamma)V^{G(N-1)}]; \]

3). Banks break even:

\[ \pi R_l \geq R_d; \]

4). A constant steady state population distribution:

\[ \alpha^{G(k-1)}(1 - c)p(1 - \pi) = \alpha^{G(k)}(1 - c)p\pi \quad k = 1, 2, \ldots N - 2, \]

\[ \alpha^{G(N-1)}(1 - c)p(1 - \pi)\gamma = (1 - \sum_{k=1}^{N-1} \alpha^{G(k)})\eta, \]

where \( \alpha^{G(k)} \) denotes the proportion of agents with rating \( k \) \((k = 1, 2, \ldots N - 1)\).

Same as before, we assume that competition drives the deposit rate and the loan rate to the minimum level; that is, \( R_d = X_L \) and \( \pi R_l = X_L \). It is trivial to see that the expected lifetime value decreases as an agent’s rating deteriorates. As a matter of fact, the rating
system creates a chain of incentive constraints that gives every borrower a carrot-and-stick choice: a rating upgrade for repayment or a rating downgrade for default. In equilibrium, all borrowers with high outputs choose the carrot.

**Proposition 5** If an equilibrium with \( N \) ratings exists, the value functions are represented as:

\[
V_G(k) = \frac{cpX_H+c(1-p)X_L+(1-c)p(X_H-X_L)}{\beta(Y_G(k))} - Y_G(k),
\]

where \( Y_G(k) \) follows the following recursive rules:

1) for \( k = 1, 2, ..., N-1 \), \( N(E) \), \( m_1 = 0 \), and \( m_{k+1} = \frac{\beta + (1-c)\pi \cdot m_k}{\beta + (1-c)p(1-\pi) + (1-c)p\pi m_k} \);

2a) for agents with rating \( N(E) \), we have \( Y_G = \frac{(1-c)p(X_H-X_L)}{(\beta+\gamma)(1-c)p(1-\pi)+(1-c)p\pi m_{N-1}} \);

2b) for agents with rating \( k = N-1 \), we have \( Y_G(N-1) = \frac{\gamma(1-c)p(1-\pi)Y_G}{\beta + \gamma(1-c)p(1-\pi)+(1-c)p\pi m_{N-1}} \);

2c) for agents with rating \( k = 1, 2, ..., N-2 \), we have \( Y_G(k) = \frac{(1-c)p(1-\pi)Y_G(K+1)}{\beta + (1-c)p(1-\pi)+(1-c)p\pi m_k} \).

**Proof.** See Appendix. ■

The proposition above tells us the solutions to the value functions conditional on the existence of a steady state equilibrium, which needs to satisfy all the constraints. The following lemma simplifies the analysis and enables us to pin down the condition under which an equilibrium exists.

**Lemma 1** For \( k = 1, 2, ..., N-1 \), \( N(E) \), the incentive compatibility constraint of agents with rating \( k \) subsumes that of agents with rating \( k + 1 \).

**Proof.** See Appendix. ■

Lemma 1 essentially says that the only incentive compatibility constraint that matters is that of agents with the best rating. In other words, as an agent’s rating drops, the cost of default increases in an accelerated speed. As a result, the incentive compatibility constraint of agents with the best rating determines whether an equilibrium exists.
Proposition 6  In equilibrium, a rating system can only consist of a finite maximum number, $\tilde{N}$, of ratings, with $\tilde{N}$ determined by the incentive compatibility condition of agents with the best rating. Moreover, $\tilde{N}$ is increasing in $\gamma$ and decreasing in $\eta$. If an equilibrium with $\tilde{N}$ ratings exists, then there also exist equilibria with $2, 3, \ldots, N - 1$ ratings, but the equilibrium with $\tilde{N}$ ratings is the most efficient.

Proof. See Appendix. □

5.3 Optimal Rating System

Our analysis above shows that the allowed maximum number of ratings is increasing in the severity of the punishment imposed on defaulted borrowers with the worst rating: the chance of borrowers with the rating $N - 1$ to be excluded from borrowing, $\gamma$, and the chance of those excluded agents to be absolved and allowed to borrow again, $\eta$. Since credit exclusion precedes forgiveness and absolution, the parameter $\gamma$ plays a more important role than $\eta$.

Proposition 7  In an equilibrium with credit ratings, social welfare only depends on the ratio of $\gamma$ to $\eta$. Given the ratio $\gamma/\eta$, a greater value of $\gamma$ allows a weakly more efficient equilibrium.

Proof. See Appendix. □

Based on Proposition 7, we can set $\gamma$ equal to one and analyze the effect of $\eta$ on social welfare. On the one hand, a lower $\eta$ allows a greater number of ratings that can improve social welfare; on the other hand, a lower lower $\eta$ implies that it is more difficult for agents who are shut out of borrowing to go back. In the extreme case, when $\eta$ goes to zero, almost every agent is prohibited from borrowing and we essentially retrogress to autarky, which is the most inefficient case. Therefore, there must exist an interior solution to $\eta$ which delivers the optimal social welfare.

Proposition 8  There exists an interior $\eta^* \in (0, 1]$ that determines the optimal number of ratings and delivers the optimal social welfare.
Proof. See Appendix.

The optimal rating system is based on the trade-off between how frequently a borrower is excluded from the loan market and how long it takes for him to get back. A lower $\eta$ make it more difficult for an excluded borrower to get a fresh start; on the other hand, the more severe punishment enables the system to increase the number of ratings and give an average borrower more chances to repair his credit rating before the worst rating befalls him and shuts him out of borrowing. Credit exclusion creates the valuable soft collateral, but it is also the ultimate source of inefficiency in a world without search frictions. The optimal value of $\eta$ minimizes the social cost by minimizing the steady state population of agents who are excluded from borrowing.

6 Discussion

So far we have assumed that, if a borrower is not excluded from the capital market, then the interest rate he pays is the same regardless of his rating. This feature is different from other papers in the literature, such as Diamond (1989), Vercammen (1995), and Padilla and Pagano (2000), that use interest rate differentiation to incentivise borrowers. While these papers are based on unobservable ex ante heterogenous borrow qualities, our paper is based on the assumption that borrowers’ production shocks are public information. This assumption allows us to zero in on the disciplinary effect of credit ratings. We will show that, in our framework, interest rate differentiation cannot achieve the same disciplinary effect as credit exclusion does.

Considering the three-tier rating system we solved in Section 5.1, now instead of the different threats of credit exclusion, we assume that borrowers with different ratings are charged with different interest rates: $R_t^A$ and $R_t^B$ for borrowings with ratings $A$ and $B$ respectively. Agents with rating $A$ have the following value functions once the capital and
productivity shocks are realized:

\[
V^{RA}_{1H} = \frac{1}{1 + \beta} \{X_H + V^{RA}\},
\]
\[
V^{RA}_{1L} = \frac{1}{1 + \beta} \{R_d + V^{RA}\},
\]
\[
V^{RA}_{0H} = \frac{1}{1 + \beta} \{\pi(X - R_i^A + V^{RA}) + (1 - \pi)V^{RB}\},
\]
\[
V^{RB}_{0L} = \frac{V^{RA}}{1 + \beta},
\]

where

\[
V^{RA} \equiv cpV^{RA}_{1H} + c(1 - p)V^{RA}_{1L} + (1 - c)pV^{RA}_{0H} + (1 - c)(1 - p)V^{RA}_{0L}
\]

is the unconditional expected lifetime value at the beginning of a period. Agents with rating \(B\) have the following value functions once the capital and productivity shocks are realized:

\[
V^{RB}_{1H} = \frac{1}{1 + \beta} \{X_H + V^{RB}\},
\]
\[
V^{RB}_{1L} = \frac{1}{1 + \beta} \{R_d + V^{RB}\},
\]
\[
V^{RB}_{0H} = \frac{1}{1 + \beta} \{\pi(X - R_i^B + V^{RA}) + (1 - \pi)V^{RB}\},
\]
\[
V^{RB}_{0L} = \frac{V^{RB}}{1 + \beta},
\]

where

\[
V^{RB} \equiv cpV^{RB}_{1H} + c(1 - p)V^{RB}_{1L} + (1 - c)pV^{RB}_{0H} + (1 - c)(1 - p)V^{RB}_{0L}
\]

is the unconditional expected lifetime value at the beginning of a period.

The value functions are subject to the following constraints:

1). Depositors are willing to deposit their capital in banks:

\[
R_d \geq X_L;
\]
2). Borrowers with ratings $A$ and $B$ are willing to repay bank loans:

$$R_t^A \leq R_t^B \leq V^{RA} - V^{RB};$$

3). Banks break even:

$$\pi[pR_t^A + (1 - p)R_t^B] \geq R_d;$$

where $p$ and $(1-p)$ are the fractions of borrowers with ratings $A$ and $B$ respectively, because none of the agents is excluded from the market.

Same as before, we assume that competition drives the deposit rate and the loan rate to the minimum level; that is, $R_d = X_L$ and $\pi[pR_t^A + (1 - p)R_t^B] = X_L$.

**Proposition 9** There does not exist a steady state equilibrium where borrowers with different ratings are charged with different interest rates.

**Proof.** See Appendix. ■

The reason why interest rate differentiation cannot support a steady state equilibrium is that the value functions are endogenized. When the interest rates are the same ($R_t^A = R_t^B$), the expected lifetime values are also the same ($V^{RA} = V^{RB}$). As we increase the difference between $R_t^A$ and $R_t^B$, the difference between $V^{RA}$ and $V^{RB}$ also increases. However, the difference between the two expected lifetime values does not increase as fast as the difference between the two interest rates because future production shocks are independent of ratings. As a result, interest rate differentiation cannot satisfy the incentive compatibility conditions. To make the rating system work, credit exclusion is indispensable.

7 Conclusion

In this paper, we show that exclusion from the capital market is a form of soft collateral that can be used to alleviate a borrower’s incentive to default strategically. This soft collateral is
very weak in a dispersed individual loan market because search frictions not only decrease the probability of meeting a lender, but increase the difficulty of sharing information about defaulted loans. Banks arise to improve efficiency in the sense that a centralized market facilitates borrowing, makes it easier to identify a defaulted borrower, and boosts the value of the soft collateral. A credit rating system can further improve efficiency because stratified ratings make it possible to fine-tune borrowers’ incentives.

The results we produce are consistent with the ways in which credit ratings work in the real world. There is a steady state distribution of credit ratings at any given time; agents’ ratings migrate over time; interest rate differentiation cannot substitute for credit exclusion to discipline borrowers; excluded borrowers are allowed to return to the capital market; and credit ratings are coarse.
References


Economy 97, 828-862.


Downside of Precise Credit Ratings.” London School of Economics working paper.


[8] Fulghieri, Paolo, Gunter Strobl, and Han Xia (2014). “The Economics of Solicited and
Unsolicited Credit Ratings.” Review of Financial Studies 27, 484-518.

University working paper.

ton Working paper.

nomics and Finance, working paper.


Appendix

**Proof of Proposition 1:** Plugging $V_{1H}^A, V_{1L}^A, V_{0H}^A,$ and $V_{0L}^A$ into the expression for $V^A$, we can easily get:

\[
V^A \equiv cpV_{1H}^A + c(1-p)V_{1L}^A + (1-c)pV_{0H}^A + (1-c)(1-p)V_{0L}^A
\]

\[
= \frac{1}{1+\beta} \{ cpX_H + c(1-p)X_L + V^A \}.
\]

The solutions are:

\[
V^A = \frac{cpX_H + c(1-p)X_L}{\beta},
\]

\[
V_{0H}^A = V_{0L}^A = \frac{cpX_H + c(1-p)X_L}{(1+\beta)\beta},
\]

\[
V_{1L}^A = \frac{cpX_H + c(1-p)X_L}{(1+\beta)\beta} + \frac{X_L}{1+\beta},
\]

\[
V_{1H}^A = \frac{cpX_H + (1-p)X_L}{(1+\beta)\beta} + \frac{X_H}{1+\beta}.
\]

**Proof of Proposition 2:** Plugging $\pi R = X_L, V_{1H}^I, V_{1L}^I, V_{0H}^I,$ and $V_{0L}^I$ into the expression for $V^I$, we have:

\[
[\beta + c(1-c)p(1-p)(1-\pi)\gamma]V^I = cpX_H + c(1-p)X_L + c(1-c)p(1-p)(X_H - X_L)
\]

\[
+c(1-c)p(1-p)(1-\pi)\gamma V_e^I.
\]

In combination with the equation for $V_e^I$, we can get:

\[
V^I = \frac{cpX_H + c(1-p)X_L}{\beta} + \frac{\beta + \eta}{\beta}[\beta + \eta + c(1-c)p(1-p)(1-\pi)\gamma]V^I,
\]

\[
V_e^I = \frac{cpX_H + c(1-p)X_L}{\beta} + \frac{\eta}{\beta}[\beta + \eta + c(1-c)p(1-p)(1-\pi)\gamma]V_e^I.
\]
The borrower’s incentive compatibility condition is:

\[
\frac{X_L}{\pi} \leq \gamma (V^I - V_e^I) = \frac{[c(1-c)p(1-p)\pi](X_H - X_L)}{\frac{\beta + \eta}{\gamma} + c(1-c)p(1-p)(1-\pi)}.
\]

The incentive compatibility condition is satisfied if \( \frac{\beta + \eta}{\gamma} \) is small enough; that is, given all other variables, the existence of a private loan equilibrium requires a minimum (or maximum) value of \( \gamma \) (or \( \eta \)), denoted by \( \gamma^I \) (or \( \eta^I \)).

Social welfare in this case is equal to the weighted average of the expected lifetime value: \( W^I = \alpha^I V^I + (1 - \alpha^I) V_e^I \). Plugging in the steady state population distribution: \( \alpha^I = \frac{1}{1+ (c(1-c)p(1-p)(1-\pi)\gamma/\eta)} \), we have:

\[
W^I = \frac{cpx_H + c(1-p)x_L}{\beta} + \frac{c(1-c)p(1-p)(X_H - X_L)}{\beta[1 + c(1-c)p(1-p)(1-\pi)\gamma/\eta]}.
\]

**Proof of Proposition 3:** We assume the loan rate is set at the minimum; that is, \( R_d = X_L \) and \( \pi R_l = X_L \). Similar to the proof of Proposition 2, we can get:

\[
V^B = \frac{cpx_H + c(1-p)x_L}{\beta} + \frac{(\beta + \eta)(1-c)p\pi(X_H - X_L)}{\beta[\beta + \eta + (1-c)p(1-\pi)\gamma]}
\]

\[
V_e^B = \frac{cpx_H + c(1-p)x_L}{\beta + \eta} + \frac{\eta(1-c)p\pi(X_H - X_L)}{\beta[\beta + \eta + (1-c)p(1-\pi)\gamma]}.
\]

The borrower’s incentive compatibility condition can be simplified as:

\[
\frac{X_L}{\pi} \leq \frac{(1-c)p(X_H - X_L)}{\frac{\beta + \eta}{\gamma} + (1-c)p(1-\pi)}.
\]

The incentive compatibility condition is satisfied if \( \frac{\beta + \eta}{\gamma} \) is small enough. It is trivial to see the required minimum value of \( \frac{\beta + \eta}{\gamma} \) is greater than that for the existence of a private loan equilibrium, which implies relaxed cutoff values for \( \gamma \) and \( \eta \): \( \gamma^I > \gamma^B \); \( \eta^I < \eta^B \).
Social welfare in the bank loan equilibrium is equal to the weighted average of the expected lifetime value: \( W^B = \alpha^B V^B + (1 - \alpha^B) V_e^B \). Plugging in the steady state population distribution: \( \alpha^B = \frac{1}{1 + (1 - c)p(1 - \pi)\gamma/\eta} \), we have:

\[
W^B = \frac{cpX_H + c(1 - p)X_L}{\beta} + \frac{(1 - c)p(X_H - X_L)}{\beta[1 + (1 - c)p(1 - \pi)\gamma/\eta]}.
\]

It is trivial to see that a bank loan equilibrium dominates a private loan equilibrium.

**Proof of Proposition 4:** We can reduce the equilibrium to the following three equations:

\[
[\beta + (1 - c)p(1 - \pi)] V^{RA} = cpX_H + c(1 - p)X_L + (1 - c)p(X_H - X_L) + (1 - c)p(1 - \pi)V^{RB},
\]

\[
[\beta + (1 - c)p\pi + (1 - c)p(1 - \pi)\gamma] V^{RB} = cpX_H + c(1 - p)X_L + (1 - c)p(X_H - X_L) + (1 - c)p\pi V^{RA} + (1 - c)p(1 - \pi)\gamma V_e^I,
\]

\[
[\beta + \eta] V_e^R = cpX_H + c(1 - p)X_L + \eta V^{RB}.
\]

The solutions are:

\[
V^{RA} = \frac{cpX_H + c(1 - p)X_L}{\beta} + \frac{(1 - c)p(X_H - X_L)}{\beta \{1 + \frac{[\beta + (1 - c)p(1 - \pi)](1 - c)p(1 - \pi)\gamma}{(\beta + \eta)[\beta + (1 - c)p]} \}},
\]

\[
V^{RB} = \frac{cpX_H + c(1 - p)X_L}{\beta} + \frac{(1 - c)p(X_H - X_L)}{\beta \{1 + \frac{[\beta + (1 - c)p(1 - \pi)](1 - c)p(1 - \pi)\gamma}{(\beta + \eta)[\beta + (1 - c)p]} \}},
\]

\[
V_e^R = \frac{cpX_H + c(1 - p)X_L}{\beta} + \frac{\eta(1 - c)p(X_H - X_L)}{\beta(\beta + \eta) \{1 + \frac{[\beta + (1 - c)p(1 - \pi)](1 - c)p(1 - \pi)\gamma}{(\beta + \eta)[\beta + (1 - c)p]} \}}.
\]

Because \( V_e^R < V^{RB} \), the incentive compatibility constraint of agents with rating \( A \) implies that of agents with rating \( B \). Hence the existence of a steady state equilibrium requires:

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\[
\frac{X_L}{\pi} \leq V^{RA} - V^{RB} = \frac{(1 - c)p(X_H - X_L)}{\frac{[\beta + (1-c)p(\beta+n)]}{(1-c)p(1-\pi)^\gamma} + [\beta + (1 - c)p(1 - \pi)]}.
\]

Since \(\frac{[\beta + (1-c)p]}{(1-c)p(1-\pi)} > 1\) and \(\beta > 0\), the required value of \(\frac{(\beta+n)}{\gamma}\) for the existence of a steady state equilibrium is smaller than that for the existence of a bank loan equilibrium without credit rating; that is, \(\gamma^R > \gamma^B\); \(\eta^R < \eta^B\).

The steady state equilibrium population distribution is as follows:

\[
\alpha^{RA} = \frac{\pi}{1 + (1 - c)p(1 - \pi)^2/\eta}, \\
\alpha^{RB} = \frac{1 - \pi}{1 + (1 - c)p(1 - \pi)^2/\eta}.
\]

Social welfare is equal to:

\[
W^R = \alpha^{RA}V^{RA} + \alpha^{RB}V^{RB} + (1 - \alpha^{RA} - \alpha^{RB})V_e^R = \frac{cpX_H + c(1-p)X_L}{\beta} + \frac{(1-c)p(X_H - X_L)}{\beta[1 + (1-c)p(1 - \pi)^2/\eta]}.
\]

Compared with social welfare in a bank loan equilibrium without credit rating \(W^B\), it is trivial to see that credit rating improves efficiency so long as it exists in equilibrium.

**Proof of Proposition 5:** For \(k = 1, 2, ..., N - 1, N(E)\), defining

\[
Y^{G(k)} = \frac{cpX_H + c(1-p)X_L + (1-c)p(X_H - X_L)}{\beta} - V^{G(k)};
\]

We can transform the equations of value functions into the following expressions: for \(k = 1,\)

\[
[\beta + (1 - c)p(1 - \pi)]Y^{G(1)} = (1-c)p(1-\pi)Y^{G(2)};
\]
for $k = 2, 3, \ldots N - 2$,

$$[(\beta + (1 - c)\pi + (1 - c)p(1 - \pi))Y^{G(k)}] = (1 - c)p\pi Y^{G(k-1)} + (1 - c)p(1 - \pi)Y^{G(k+1)};$$

for $k = N - 1$,

$$[(\beta + (1 - c)\pi + (1 - c)p(1 - \pi)\gamma)Y^{G(N-1)}] = (1 - c)p\pi Y^{G(N-2)} + (1 - c)p(1 - \pi)\gamma Y_e^G;$$

and for $k = N(E)$,

$$(\beta + \eta)Y_e^G = \eta Y^{G(N-1)} - (1 - c)p(X_H - X_L).$$

We have

$$Y^{G(1)} = \frac{(1 - c)p(1 - \pi)Y^{G(2)}}{\beta + (1 - c)p(1 - \pi)}$$

and

$$Y^{G(k)} = \frac{(1 - c)p(1 - \pi)Y^{G(k+1)}}{\beta + (1 - c)p(1 - \pi) + (1 - c)p\pi m_k},$$

which implies

$$1 - m_{k+1} = \frac{(1 - c)p(1 - \pi)}{\beta + (1 - c)p(1 - \pi) + (1 - c)p\pi m_k}, \quad \text{or}$$

$$m_{k+1} = \frac{\beta + (1 - c)p\pi m_k}{\beta + (1 - c)p(1 - \pi) + (1 - c)p\pi m_k}. $$

For agents with rating $N - 1$, we have

$$[(\beta + (1 - c)\pi + (1 - c)p(1 - \pi)\gamma)Y^{G(N-1)}] = (1 - c)p\pi Y^{G(N-2)} + (1 - c)p(1 - \pi)\gamma Y_e^G,$$

or
\[
[(\beta + (1 - c)p\pi + (1 - c)p(1 - \pi)\gamma)]Y^{G(N-1)} = \frac{(1-c)^2p^2\pi(1-\pi)Y^{G(N-1)}}{\beta + (1-c)p(1-\pi) + (1-c)p\pi m_{N-2}} + (1-c)p(1-\pi)\gamma Y^G_e,
\]

which gives us
\[
Y^{G(N-1)} = \frac{(1-c)p(1-\pi)\gamma Y^G_e}{\beta + (1-c)p(1-\pi) + (1-c)p\pi m_{N-1}}.
\]

Essentially we have a series of difference equations, which can be solved using induction.

The solutions are:

1) for \(k = 1, 2, ..., N - 1, N(E), m_1 = 0, \text{ and } m_{k+1} = \frac{\beta+(1-c)p\pi m_k}{\beta+(1-c)p(1-\pi)+(1-c)p\pi m_{N-1}};\)

2a) for agents with rating \(N(E)\), we have \(Y^G_e = \frac{(1-c)p(X_H-X_L)}{\beta+(1-c)p(1-\pi)+(1-c)p\pi m_{N-1}};\)

2b) for agents with rating \(k = N - 1\), we have \(Y^{G(N-1)} = \frac{(1-c)p(1-\pi)\gamma Y^G_e}{\beta+(1-c)p(1-\pi)+(1-c)p\pi m_{N-1}};\)

2c) for agents with rating \(k = 1, 2, ..., N - 2\), we have \(Y^G(k) = \frac{(1-c)p(1-\pi)Y^{G(k+1)}}{\beta+(1-c)p(1-\pi)+(1-c)p\pi m_k} .\)

**Proof of Lemma1:** We can rewrite the incentive compatibility conditions as follows:

2a). for borrowers with rating 1:

\[R_l \leq Y^{G(2)} - Y^{G(1)};\]

2b). for borrowers with rating \(k \ (k = 2, 3, ..., N - 2):\)

\[R_l \leq Y^{G(k+1)} - Y^{G(k-1)};\]

2c). for borrowers with rating \(N - 1):\)

\[R_l \leq \gamma Y^G_e + (1-\gamma)Y^{G(N-1)} - Y^{G(N-2)} .\]

We first show that the incentive compatibility condition of borrowers with rating \(k + 1\)
subsumes that of borrowers with rating $k$ for $k = 2, 3, \ldots, N - 3$:

$$\begin{align*}
&\left[ Y^{G(k+2)} - Y^{G(k)} \right] - \left[ Y^{G(k+1)} - Y^{G(k-1)} \right] \\
= \frac{[\beta + (1 - c) p \pi m_{k+1}] Y^{G(k+2)}}{\beta + (1 - c) p (1 - \pi) + (1 - c) p \pi m_{k+1}} - \frac{[\beta + (1 - c) p \pi m_{k-1}] Y^{G(k)}}{\beta + (1 - c) p (1 - \pi) + (1 - c) p \pi m_{k-1}} \\
> 0;
\end{align*}$$

the last step results from both $Y^{G(k)}$ and $m_k$ being positive and increasing in $k$.

It is trivial to see that the incentive compatibility condition of borrowers with rating 2 implies that of borrowers with rating 1. As for borrowers with rating $N - 1$, we can easily show that the incentive compatibility condition implies that of borrowers with rating $N - 2$:

$$\begin{align*}
&\left[ \gamma Y^{G_e} + (1 - \gamma) Y^{G(N-1)} - Y^{G(N-2)} \right] \\
> Y^{G(N-1)} - Y^{G(N-2)} \\
> Y^{G(N-1)} - Y^{G(N-3)}.
\end{align*}$$

Hence the only incentive compatibility condition that matters is that of borrowers with the best rating, 1.

**Proof of Proposition 6:** Since the expected lifetime payoff of agents with the best rating, $V^{G(1)}$, is capped by $\frac{c p X_H + c (1 - p) X_L + (1 - c) p (X_H - X_L)}{\beta}$, which is achieved is they would never be excluded from borrowing, and the the expected lifetime payoff of agents excluded from borrowing, $V^{G}_e$, is floored by $\frac{c p X_H + c (1 - p) X_L}{\beta}$, which is the autarky value, the difference between these two values is finite and can only support a finite number of incentive compatibility conditions. Consequently, a rating system can only have a finite maximum number of ratings.

Following Lemma 1, the only incentive compatibility condition that matters is that of
agents with the best rating, which can be expressed as:

\[ X_L/\pi \leq Y^{G(2)} - Y^{G(1)} = \frac{\beta}{(1-c)p(1-\pi)} Y^{G(1)}. \]

In order to prove that the allowed maximum number, \( \widehat{N} \), of ratings, is increasing in \( \gamma \) and decreasing in \( \eta \), we only need to show that \( Y^{G(1)} \) is increasing in \( \gamma \) and decreasing in \( \eta \). As Proposition 5 shows that, \( Y^{G(1)} \) depends on the \( m_k \)'s and \( Y^{G(N-1)} \). Because \( m_k \)'s do not depend on \( \gamma \) or \( \eta \), it boils down to showing that \( Y^{G(N-1)} \) is increasing in \( \gamma \) and decreasing in \( \eta \), as shown below:

\[
Y^{G(N-1)} = \frac{\gamma(1-c)p(1-\pi)Y^{G}_e}{\beta + \gamma(1-c)p(1-\pi) + (1-c)p\pi \cdot m_{N-1}} \cdot \frac{(1-c)p(X_H - X_L)}{\beta + \gamma(1-c)p(1-\pi) + (1-c)p\pi \cdot m_{N-1}} \cdot \frac{\eta(1-c)p(1-\pi)}{\beta + \gamma(1-c)p(1-\pi) + (1-c)p\pi \cdot m_{N-1}}
\]

\[
= \frac{(1-c)^2p^2(1-\pi)(X_H - X_L)}{\beta + (1-c)p\pi \cdot m_{N-1}} + \beta(1-c)p(1-\pi).
\]

Next, we show that if an equilibrium with \( N \) ratings exists, then there also exists an equilibrium with \( N-1 \) ratings; hence, by induction, there exist equilibria with \( 2, 3, \ldots, N-1 \) ratings. Suffice it to show that \( Y^{G(1)} \) in a system with \( N \) ratings is smaller than \( Y^{G(1)} \) in a system with \( N-1 \) ratings. Compared a system with \( N \) ratings, the value functions in a system with \( N-1 \) ratings depends on the exact same series of \( m_k \)'s except that is truncated at \( m_{N-2} \) because \( N-2 \) is the rating next to the last rating, that is, \( N-1(E) \). Because \( m_{N-2} < m_{N-1} \), it is trivial to see that \( Y^{G(N-2)} \) in a system with \( N \) ratings is smaller than \( Y^{G(N-2)} \) in a system with \( N-1 \) ratings. As a result, \( Y^{G(1)} \) in a system with \( N \) ratings is also smaller than \( Y^{G(1)} \) in a system with \( N-1 \) ratings.

Finally, we examine social welfare, which depends on the steady state distribution of agents with different ratings. Because we have \( \alpha^{G(k)} = \frac{\pi}{1-\pi} \alpha^{G(k-1)} \), the fraction of agents with rating \( E \) is:

\[
1 - \sum_{k=1}^{N-1} \alpha^{G(k)} = 1 - \alpha^{G(N-1)} \frac{1 - \left(\frac{\pi}{1-\pi}\right)^{N-1}}{1 - \frac{\pi}{1-\pi}}.
\]
Using the steady state equilibrium condition

\[ \alpha^{G(N-1)}(1-c)p(1-\pi)\gamma = (1 - \sum_{k=1}^{N-1} \alpha^{G(k)})\eta, \]

we can get

\[ \alpha^{G(N-1)} = \frac{1}{1 - \frac{(1-\gamma)N-1}{1-\gamma}} + (1-c)p(1-\pi)\gamma/\eta. \]

So the fraction of agents with rating \( E \) is equal to

\[ 1 - \sum_{k=1}^{N-1} \alpha^{G(k)} = \frac{(1-c)p(1-\pi)\gamma/\eta}{1 - \frac{(1-\gamma)N-1}{1-\gamma}} + (1-c)p(1-\pi)\gamma/\eta. \]

Social welfare is the first best solution, \( \frac{c p X_H + c (1-p) X_L + (1-c)p(X_H - X_L)}{\beta} \) (no agent is excluded from borrowing), minus the loss due to agents with rating \( E \) being excluded:

\[ W^G = \frac{c p X_H + c (1-p) X_L + (1-c)p(X_H - X_L)}{\beta} - \frac{(1 - \sum_{k=1}^{N-1} \alpha^{G(k)})(1-c)p(X_H - X_L)}{\beta}. \]

Since \( 1 - \sum_{k=1}^{N-1} \alpha^{G(k)} \) is decreasing in \( N \) given the primitive parameters, the rating system with the maximum number of ratings, \( \hat{N} \), is the most efficient.

Proof of Proposition 7: Proposition 6 shows that social welfare is only affected by \( \gamma/\eta \).

On the other hand, the incentive compatibility condition only depends on \( Y^{G(1)} \), which can be obtained through \( Y^{G(N-1)} \) and \( m_k \)'s. \( Y^{G(N-1)} \) is related to \( \gamma \) and \( \eta \) through the term \( \frac{\beta + n}{\gamma} \), while \( m_k \)'s are not affected by either \( \gamma \) or \( \eta \). Hence if we maintain the value of \( \gamma/\eta \) and increase \( \gamma \), we can achieve the same social welfare and relax the incentive compatibility constraint; in addition, if the increase in \( \gamma \) allows us to add one more tier of rating, social welfare can be improved.

Proof of Proposition 8: Proposition 6 shows that there is exits a finite maximum
number of ratings, \( \hat{N} \), and \( \tilde{N} \) is decreasing in \( \eta \). As \( \eta \) goes to zero, the number of ratings converges to its maximum; however, social welfare retrogresses to the case of autarky because almost every agent is excluded from borrowing. So an optimal rating system is not the one that pushes the number of ratings to the maximum.

**Proof of Proposition 9:** We can reduce the equilibrium to the following three equations:

\[
[\beta + (1 - c)p(1 - \pi)]V^{RA} = cpX_H + c(1 - p)X_L + (1 - c)p(X_H - \pi R_t^A) \\
+ (1 - c)p(1 - \pi)V^{RB},
\]

\[
[\beta + (1 - c)p\pi + (1 - c)p(1 - \pi)]V^{RB} = cpX_H + c(1 - p)X_L + (1 - c)p(X_H - \pi R_t^B) \\
+ (1 - c)p\pi V^{RA},
\]

which gives us:

\[
(1 - c)p(\pi R_t^B - \pi R_t^A) = [\beta + (1 - c)p](V^{RA} - V^{RB}).
\]

It can be seen immediately that it contradicts the incentive compatibility condition \( R_t^A \leq R_t^B \leq V^{RA} - V^{RB} \).